Quantum state tomography from sequential measurement of two variables in a single setup

Antonio Di Lorenzo

Instituto de Física, Universidade Federal de Uberlândia, 38400-902 Uberlândia, Minas Gerais, Brazil

and

CNR-IMM-UOS Catania (Università), Consiglio Nazionale delle Ricerche, Via Santa Sofia 64, 95123 Catania, Italy

Abstract

We demonstrate that the task of determining an unknown quantum state can be accomplished efficiently by making a sequential measurement of two observables $\hat{A}$ and $\hat{B}$, provided that the two observables are chosen in such a way that their eigenstates may form bases connected by a discrete Fourier transform. The state can be pure or mixed, the dimension of the Hilbert space and the coupling strength are arbitrary, and the experimental setup is fixed. The concept of Moyal quasicharacteristic function is introduced for finite-dimensional Hilbert spaces.

Introduction.—A colleague has challenged you: she has built a black box from which, upon the pressing of a button, a quantum system is released. What is the state of the system? You are not allowed to open the box, nor to measure any of its properties. You can only measure the quantum system, and repeat as many times as you want. This is the essence of quantum state tomography.

The preparation of a quantum system is characterized by a quantum state, which is given by the density operator, a positive-definite operator of trace one in a Hilbert space. Often, some information about the system is missing, but it could be recovered, in principle, from the environment and from the preparing apparatus. When all this information is retrieved, which can be done without disturbing the system in any way, the quantum system is described by a pure state, i.e. a density operator of rank one, which can be written as $\rho_{\rm sys} = |\psi\rangle\langle\psi|$ in terms of a vector $|\psi\rangle$ of the Hilbert space. However, in general, this information is lost for all practical purposes, and the system is to be described by a density operator. A fundamental question is then, how do we determine the unknown state $\rho_{\rm sys}$ of a quantum system? Answering this question is the goal of quantum state tomography.

Reconstructing an unknown quantum state $\rho_{\rm sys}$ is believed to be a difficult task, requiring the separate measurement of several observables. The usual approach is to take the system in the unknown state and measure the statistics of an observable $\hat{A}_1$, then, with a distinct ensemble of identically prepared systems, measure another observable $\hat{A}_2$, etc. The observables $\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_n$ needed to reconstruct the quantum state are known as the *quorum*, and they usually number as d^2, with d the dimension of the Hilbert space, even though some improvement over this number can be achieved [1]. Usually, from each measurement, only the average value is extracted. For instance, to reconstruct the state of a spin 1/2 system, the average values $n_j = \langle\sigma_j\rangle$, $j = x, y, z$, are calculated, and the state $\rho_{\rm sys} = (1 + \mathbf{n}\cdot\boldsymbol{\sigma})/2$ is reconstructed. The noise introduced by the detectors is then a hindrance.

Furthermore, the most commonly used statistical tool for the reconstruction of the state is the maximum likelihood estimation, which does not take into account the positive-definiteness of the density operator and may give rise to rank-deficient estimates. *Ad hoc* corrections are often devised to overcome this difficulty. The recently introduced Bayesian [2] approach has solved this last issue, but its adoption is being slow. We remark that in the Bayesian approach, the maximum likelihood estimate is justified when uniform priors are assumed and a particular cost function is postulated [3]. In any case, the number of

different setups needed for quantum state tomography increases with the dimension of the Hilbert space, making the process time-consuming.

Recently, many schemes based on weak measurement [4–8] have been proposed for quantum state tomography. However, a distinct disadvantage of such schemes is that on one hand the formulas for the weak measurement are approximated, introducing a further uncertainty in the reconstruction, and on the other hand the weak measurement relies on postselection, which requires that only a fraction of the data is retained, yielding a reduced efficiency.

Haapasalo *et al.* [9] have also pointed out the superiority of phase space methods over the weak measurement methods in order to reconstruct the wave-function. This suggests to look for an extension of phase space methods to finite dimensional Hilbert spaces. In doing so, we shall propose a generalization of the Wigner [10] and the Moyal [11] function. The justification for this choice is that the Moyal function has revealed itself to be an extremely useful tool for describing the statistics of joint and sequential measurements of momentum and position [12, 13].

A promising avenue for efficient quantum state tomography was opened by considering measurements in mutually unbiased bases [14, 15]. All the proposals of which we are aware, however, require many different setups, at least as many as the dimension of the Hilbert space.

Here, instead, we propose a quantum state tomography scheme consisting in a *single* sequential measurement of *arbitrary strength* and relying on an *exact relation* between the initial state of the system and the final output of the measurement. The whole statistics of the measurement is used, and the unsharpness of the detector is turned into a resource, rather than an obstacle. Our scheme uses a particular pair of mutually unbiased bases, the Fourier conjugated bases. We demonstrate that there are infinitely many pairs of observables $\hat{A}, \hat{B}$ that allow the reconstruction of an unknown quantum state ρ_{sys}, be this pure or mixed. Furthermore, by suitably choosing the first measured observable $\hat{A}$, it is possible to obtain the representation of the state, $\langle m|\rho_{\text{sys}}|m'\rangle$, in any basis of choice. We recover the result of Ref. [13] in the limit $d \to \infty$.

Main result.—In a Hilbert space of arbitrary dimension $d = 2S + 1$, there are infinitely many pairs of complementary (see below for a definition of complementary operators in a finite-dimensional Hilbert space, following Schwinger [16]) operators $\hat{A}$ and $\hat{B}$ such that any unknown quantum state ρ_{sys} can be reconstructed from the joint probability of observing the

readout values J_A and J_B in two probes that measure the two observables in sequence. For brevity, in the following we indicate with $J = [J_A, J_B]$, $\phi = [\phi_A, \phi_B]$ vectors in an auxiliary two-dimensional Euclidean space, and with $J \cdot \phi$ their scalar product.

Precisely, let $\hat{A}$, $\hat{B}$ two operators whose bases form a Fourier conjugated pair, defined below, and whose spectra are spaced by $1/\sqrt{d}$ and $2\pi/\sqrt{d}$, respectively; let $\mathcal{P}(J_A, J_B)$ the joint probability of observing the outputs J_A, J_B in two probes that make a nondemolition measurement of the system; let $z_0(\phi)$ the characteristic function describing the smearing of the readout states (i.e., even if the actual value of the pointer is J, the readout turns out to be μ with a probability $p_0(\mu - J)$); let $\mathcal{Z}(\phi_A, \phi_B)$ the Fourier transform of $\mathcal{P}(J_A, J_B)$; let

$$\mathcal{M}_{\rm sys}(\phi_A; a) = \sum_{A \in D_a} e^{i\phi_A A} \langle A + \frac{a}{2} | \rho_{\rm sys} | A - \frac{a}{2} \rangle, \tag{1}$$

the initial Moyal quasicharacteristic function of the system relative to the basis of the eigenstates of $\hat{A}$; let $\rho_{\rm pr}$ the initial state of the two probes, which is supposed to be known; let

$$\mathcal{M}_{\rm pr}(\phi; j) = \int dJ \, e^{i\phi \cdot J} \langle J + \frac{j}{2} | \, \rho_{\rm pr} | J - \frac{j}{2} \rangle, \tag{2}$$

the initial Moyal function of the probes. Then, the following relation holds between the final characteristic function and the initial Moyal functions

$$\mathcal{Z}(\phi) = z_0(\phi) \Big[\mathcal{M}_{\rm pr}(\phi; -\sigma_+ \phi) \mathcal{M}_{\rm sys}(\phi_A; \phi_B) + \mathcal{M}_{\rm pr}(\phi; -\sigma_+ \bar{\phi}) \mathcal{M}_{\rm sys}(\phi_A; \bar{\phi}_B) \Big], \tag{3}$$

for any ϕ_A and for $\phi_B = n/\sqrt{d}$, with n an integer in the range $[1-d, d-1]$, excluding $n = 0$; here, $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $\bar{\phi}_B = [n - \mathrm{sgn}(n) d]/\sqrt{d}$. Notice that $\bar{\bar{\phi}}_B = \phi_B$. Equation (3) is the central result of this paper. Starting from it, one estimates $\mathcal{Z}(\phi)$ at the discrete points $\phi_B = n_B/\sqrt{d}$ and $\bar{\phi}_B = [n_B - \mathrm{sgn}(n_B) d]/\sqrt{d}$, for fixed $\phi_A = 2\pi n_A/\sqrt{d}$, solves the two linear equations in the unknowns $x = \mathcal{M}_{\rm sys}(\phi_A; \phi_B)$ and $y = \mathcal{M}_{\rm sys}(\phi_A; \bar{\phi}_B)$, repeats for a different n_A, and finally reconstructs the density matrix in the basis of the eigenstate of $\hat{A}$ by inverting Eq. (1)

$$\langle A + \frac{a}{2} | \rho_{\rm sys} | A - \frac{a}{2} \rangle = \sum_{\phi_A} \frac{1}{d} e^{-i\phi_A A} \mathcal{M}_{\rm sys}(\phi_A; \phi_B = a). \tag{4}$$

In the limit $d \to \infty$, the second addend in Eq. (3) goes to zero, and the result of Ref. [13] is then recovered.

Proof.—Let us consider the probability of observing a readout $\mu = [\mu_A, \mu_B]$ from the two detectors after they have interacted with the system. In the following the index v stands for A or B, while $C = [A, B]$ is an auxiliary vector. By the generalized Born's rule,

$$P(\mu) = \mathrm{Tr}\left\{\left[\mathbb{1} \otimes \hat{F}(\mu)\right] U_{\mathrm{int}} \left[\rho_{\mathrm{sys}} \otimes \rho_{\mathrm{pr}}\right] U_{\mathrm{int}}^{\dagger}\right\} \tag{5}$$

with U_{int} time evolution operator and $\hat{F}(\mu) = \hat{F}_A(\mu_A) \otimes \hat{F}_B(\mu_B)$ the readout states of the probes. These are non-negative operators that account for the probes being macroscopic objects, of which some thermodynamic properties μ are observed, while their exact microstate is unknown. Furthermore, while the readout μ is ideally a function of the microscopic variables (for typically, an average or a sum, depending whether we are observing an intensive or extensive quantity), we account for some imperfections so that, while all microscopic states are equiprobable a priori, the probability for each of them to produce an outcome μ maybe different. We assume that the readout states are classical, in the sense that $[\hat{F}(\mu), \hat{F}(\mu')] = 0$, $\forall \mu, \mu'$. Thus, their common eigenstates form an orthonormal basis $|J\rangle$, where J is continuous for all practical purposes (think of J as a generalized momentum). As we are assuming sequential nondemolition measurements, $U_{\mathrm{int}} = U_B U_A$, where U_v acts on the system and the detector that measures $\hat{v}$, and there are nonentangled bases $|J_v, C_v\rangle$, such that

$$U_v |J_v, C_v\rangle = |J_v + f_v(C_v), C_v\rangle, \tag{6}$$

where $|C_v\rangle$ is an eigenstate of the observable $\hat{v}$. For a linear measurement $f_v(C_v) = \lambda_v C_v$. We assume that the interaction is engineered so that this privileged basis coincides, for the part concerning the detector, with the basis diagonalizing the readout states. A sufficient condition, for instance, is to consider the von Neumann protocol, i.e. an instantaneous interaction $H_{int} = \sum_v \delta(t - t_v) \hat{v} \hat{\Phi}_v$, which gives $U_v = \exp[i f_v(\hat{v}) \hat{\Phi}_v]$, where $\hat{\Phi}_v$ is the generator of the translations for the $|J_v\rangle$ basis.

Next, we consider the characteristic function, defined as the Fourier transform of the observable probability,

$$\mathcal{Z}(\phi) = \int d\mu e^{i\phi\cdot\mu} \mathcal{P}(\mu). \tag{7}$$

We substitute Eq. (5) into Eq. (7) and take the trace in the basis $|B, J_A, J_B\rangle$, obtaining

$$Z(\phi) = \int dJ d\mu \sum_{B, A', A''} e^{i\phi\cdot\mu} f(\mu|J) \langle B|A'\rangle \langle A', J - C'|\rho_{\mathrm{sys}} \otimes \rho_{\mathrm{pr}}|A'', J - C''\rangle \langle A''|B\rangle, \tag{8}$$

where we introduced twice the identity over the Hilbert space of the system as $\mathbb{1}_{\rm sys} = \sum_{A'} |A'\rangle\langle A'|$ and defined $C' = [A', B]$, $C'' = [A'', B]$. We assume that $f(\mu|J) = p_0(\mu - J)$, with p_0 a probability, representing the smearing out of the readout. Then, by the theorem of convolution, we have

$$\mathcal{Z}(\phi) = z_0(\phi)\mathcal{Z}_{int}(\phi) \tag{9}$$

with

$$\mathcal{Z}_{int}(\phi) = \sum_{B,A',A''} \int dJ e^{i\phi\cdot J} \langle J - C'|\rho_{\rm pr}|J - C''\rangle\langle B|A'\rangle\langle A'|\rho_{\rm sys}|A''\rangle\langle A''|B\rangle. \tag{10}$$

Now, let us define $A = (A' + A'')/2$ and $a = A' - A''$ and change the integration variables to $J_A - A$ and $J_B - B$. Then,

$$\mathcal{Z}_{int}(\phi) = \sum_{A,a} e^{i\phi_A A}\langle A - \frac{a}{2}|e^{i\phi_B \hat{B}}|A + \frac{a}{2}\rangle\langle A + \frac{a}{2}|\rho_{\rm sys}|A - \frac{a}{2}\rangle\mathcal{M}_{\rm pr}(\phi; j_a), \tag{11}$$

where $j_a = [-a, 0]$, and we introduced the Moyal quasicharacteristic function for the probes, as defined in Eq. (2). Thus,

$$\mathcal{Z}_{int}(\phi) = z_0(\phi) \sum_{a} \mathcal{M}_{\rm pr}(\phi; j_a) N_{\rm sys}(a|\phi) \tag{12}$$

where we introduced

$$N_{\rm sys}(a|\phi) = \sum_{A\in D_a} e^{i\phi_A A}\langle A - \frac{a}{2}|e^{i\phi_B \hat{B}}|A + \frac{a}{2}\rangle\langle A + \frac{a}{2}|\rho_{\rm sys}|A - \frac{a}{2}\rangle. \tag{13}$$

Notice that the domain of summation in A depends on a. In general, equations (12) and (13) are too complicated to invert and be useful in reconstructing the quantum state. For instance, if $\hat{A}$ and $\hat{B}$ commute, only diagonal terms contribute to $N_{\rm sys}(a|\phi)$, so that no reconstruction of the quantum state is possible, as one can only find the the diagonal elements of ρ_{sys}, as expected. Furthermore, if $\hat{A}$ and $\hat{B}$ have mutually unbiased bases with a constant relative phase, such that $\langle A|B\rangle = 1/\sqrt{d}$, then $N_{\rm sys}(a|\phi) = g(\phi_B)\mathcal{M}_{\rm sys}(\phi_A; a)$, with $g(\phi_B) = \sum_B \exp(i\phi_B B)/d$, and no actual simplification occurs.

On the other hand, it is clear from Eq. (13) that if for some ϕ_B the operator $\exp(i\phi_B \hat{B})$ translates the eigenstates of $\hat{A}$, then some (actually: two) Kronecker deltas in a appear. Thus, we exploit the freedom that we have in choosing the bases $|A\rangle$ and $|B\rangle$, and we assume that they are Fourier conjugated, i.e.,

$$\langle A|B\rangle = \frac{\exp[iBA]}{\sqrt{d}}. \tag{14}$$

Later on, we give more mathematical details about conjugated bases, which, to the best of my knowledge, were first introduced by Schwinger [16]. Before proceeding further, let us introduce some conventions on the indices. We shall write the dimension of the space as $d = 2S+1$. The symbol m characterizes all integers of half-integers in the range $[-S, S]$, where m inherits the property of being integer or half-integer from S. The symbol n characterizes instead an integer in the range $[-2S, 2S = d-1]$. The symbol k characterizes a number in the range $[-S + |n|/2, S - |n|/2]$ for a fixed n; as for m, depending whether $S - |n|/2$ is integer or half-integer, so is k. The numbers A are taken to be of the form $A = m/\sqrt{d}$ or $A = k/\sqrt{d}$, depending on the context, the numbers a are $n/\sqrt{d}$, and the numbers B are $2\pi m/\sqrt{d}$.

We substitute Eq. (14) into Eq. (13), obtaining

$$N_{\text{sys}}(a|\phi) = \sum_B \sum_{A=k/\sqrt{d}} \frac{e^{i\phi_A A + i(\phi_B - a)B}}{d} \langle A + \frac{a}{2}|\rho_{\text{sys}}|A - \frac{a}{2}\rangle = \frac{\sin\left[\pi(\phi_B - a)\sqrt{d}\right]}{d\sin\left[\pi(\phi_B - a)/\sqrt{d}\right]} \mathcal{M}_{\text{sys}}(\phi_A; a). \tag{15}$$

We introduced the Moyal quasicharacteristic function of the system, relative to the $|A\rangle$ basis, defined in Eq. (1). Furthermore, for $\phi_B = n/\sqrt{d}$, $N_{\text{sys}}(a|\phi)$ in Eq. (15) simplifies to

$$N_{\text{sys}}(a|\phi) = \delta_{a,\phi_B} \mathcal{M}_{\text{sys}}(\phi_A; \phi_B) + \delta_{a,\bar{\phi}_B} \mathcal{M}_{\text{sys}}(\phi_A; \bar{\phi}_B). \tag{16}$$

For $\phi_B = 0$, instead, only one term survives,

$$N_{\text{sys}}(a|[\phi_A, 0]) = \delta_{a,0} \mathcal{M}_{\text{sys}}(\phi_A; 0). \tag{17}$$

Hence, after substituting Eq. (16) into Eq. (12) evaluated at the discrete points $\phi_B = n_B/\sqrt{d}$, we get the main result Eq. (3).

Final considerations.—As both $\hat{A}$ and $\hat{B}$ have the same eigenvalues, except for a trivial rescaling, we can write $\hat{B} = 2\pi U \hat{A} U^\dagger$, with U a unitary operator. Precisely, $U = \sum_j |\widetilde{j}\rangle\langle j|$. Let us say, for definiteness, that the Hilbert space represents an angular momentum S, and that $\hat{A} = \delta a \hat{S}_z$ is proportional to an angular momentum operator, in the sense that upon rotation it transforms accordingly. The natural question arises: is $\hat{B}/\delta b$ an angular momentum operator as well? i.e., is there a unit vector $\mathbf{n}$ such that $\hat{B} = \delta b \mathbf{n} \cdot \hat{\mathbf{S}}$? The answer is no, unless $d = 2$, since in this latter case any unitary operator corresponds to a rotation. In general, however, the distinct unitary operators, modulo a global phase, are

characterized by $d^2 - 1$ real parameters, while there are only three independent rotations [17]. The proof that, for $d > 2$, none of these rotations yields $\hat{B}/\delta b = \hat{S}_{\mathbf{n}}$ is as follows: since $\exp(-(i/\sqrt{d})\hat{B})|S\rangle = \pm|-S\rangle$, $\hat{B}$ must be $\hat{B} = (2z+1)\pi\sqrt{d}\hat{S}_{\perp}$, with $z \in \mathbb{Z}$ and the $\perp$ symbol indicating an appropriate direction in the plane orthogonal to Z. Thus, $\hat{S}_{\mathbf{n}} = [(2z+1)d/2]\hat{S}_{\perp}$. This equation implies necessarily that $\perp = \pm\mathbf{n}$, $d = 2$ and either $z = 0$ or $z = -1$.

Anyhow, Reck *et al.* [18] have proved that any unitary operator U in a finite-dimensional Hilbert space can be realized by a suitable combination of elementary unitary operators that act nontrivially only in a two–dimensional subspace. Furthermore, in quantum computation, it is well known that if the Hilbert space is made up of N distinguishable qubits, any unitary operator can be approximated at will by a sequence of controlled nots and of elementary unitary operations on each qubit.

The main problem consists then in constructing the operator $\hat{A}$, in the worst case scenario that this is not provided to us by Nature. For a system composed of n distinguishable qubits, the operator $\hat{A}$ can be constructed, apart from a trivial shift and rescaling as $\hat{A} = \sum_{p=1}^{N} 2^{p-1}\sigma_{z,p}$, with $\sigma_{z,p}$ a spin operator on the p-th qubit.

I thank Giuseppe Falci for discussions. This work was performed as part of the Brazilian Instituto Nacional de Ciência e Tecnologia para a Informação Quântica (INCT–IQ) and it was partially funded by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) through process no. 245952/2012-8.

Appendix: Fourier conjugated bases

Let us consider a Hilbert space [19] of dimension d. Let $\{|m\rangle\}$ an orthonormal basis. We shall label the different elements with an index m $[-S, S]$, with $S = (d-1)/2$. We define the conjugate basis [16]

$$|\widetilde{m}\rangle = \frac{1}{\sqrt{d}} \sum_{m'} \exp[2\pi i m m'/d] |m'\rangle. \tag{18}$$

Notice that the tilde is a symbol attached to the basis, not to the index. Then, we build two operators $\hat{A}$ and $\hat{B}$, having as eigenstates the bases $|m\rangle$ and $|\widetilde{m}\rangle$, respectively, and as eigenvalues $a_m = m\delta a$, with $\delta a = 1/\sqrt{d}$, and $b_m = m\delta b$, $\delta b = 2\pi/\sqrt{d}$. In other words $\hat{A}|m\rangle = (m/\sqrt{d})|m\rangle$ and $\hat{B}|\widetilde{m}\rangle = (2\pi m/\sqrt{d})|\widetilde{m}\rangle$. In the following, we shall also indicate the elements of the first basis by means of the corresponding eigenvalue of $\hat{A}$, i.e., $|A\rangle \equiv |m\rangle$ for $A = m\delta a$, and, analogously, $|B\rangle \equiv |\widetilde{m}\rangle$ for $B = m\delta b$.

We call the operators $\hat{A}$ and $\hat{B}$ *complementary*. This terminology is justified since in the limit $d \to \infty$, the operator $l\hat{A}$ corresponds to the position operator $\hat{Q}$, while $\hbar\hat{B}/l \to \hat{P}$, with l a fundamental length scale (an educated guess: l is Planck length). In the opposite limit $d = 2$, instead, $\hat{A} = \sigma_z/2\sqrt{2}$ and $\hat{B} = -\sqrt{2}\pi\sigma_y$, with σ_j Pauli matrices.

We remark that

$$\exp[iz\delta a\hat{B}]|m\rangle = (-1)^{(d-1)r_{m-z}}|\mu(m-z)\rangle \tag{19}$$

for any m and any $z \in \mathbb{Z}$, where $\mu(l) \in [-S, S]$ and $r_l \in \mathbb{Z}$ are univocally defined by $l = r_l + \mu(l)$, for l any integer of half-integer, depending on S. In particular, $\exp[i\delta a\hat{B}]|{-S}\rangle = (-1)^{d-1}|S\rangle$ and $\exp[-i\delta a\hat{B}]|S\rangle = (-1)^{d-1}|{-S}\rangle$. Thus $\hat{B}$ is the generator of the modular translations for the basis $|A\rangle$. Conversely, $\exp[iz\delta b\hat{A}]$ are the generators of the translations in the basis $|\tilde{m}\rangle$, namely $\exp[iz\delta b\hat{A}]|\widetilde{m}\rangle = (-1)^{(d-1)r_{m+z}}|\widetilde{\mu}(m+z)\rangle$. Notice that for half-integer spin, i.e. even d, the translation comes with a sign change for odd r, and in particular $\exp[id\delta b\hat{A}] = \exp[id\delta a\hat{B}] = (-1)^{d-1}$. This is a mathematical feature of even-dimensional spaces.

[1] D. Gross, Y.-K. Liu, S. T. Flammia, S. Becker, and J. Eisert, Phys. Rev. Lett. **105**, 150401 (2010).

[2] R. Blume-Kohout, New J. Phys. **12**, 043034 (2010).

[3] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, London, 1976).

[4] H. F. Hofmann, Phys. Rev. A **81**, 012103 (2010).

[5] J. S. Lundeen, B. Sutherland, A. Patel, C. Stewart, and C. Bamber, Nature **474**, 188 (2011).

[6] J. S. Lundeen and C. Bamber, Phys. Rev. Lett. **108**, 070402 (2012).

[7] J. Fischbach and M. Freyberger, Phys. Rev. A **86**, 052110 (2012).

[8] S. Wu, Sci. Rep. **3**, 1193 (2013).

[9] E. Haapasalo, P. Lahti, and J. Schultz, Phys. Rev. A **84**, 052107 (2011).

[10] E. Wigner, Phys. Rev. **40**, 749 (1932).

[11] J. E. Moyal, Math. Proc. Cambr. Phil. Soc. **45**, 99 (1949).

[12] A. Di Lorenzo, Phys. Rev. A **83**, 042104 (2011).

[13] A. Di Lorenzo, Phys. Rev. Lett. **110**, 010404 (2013).

[14] L. M. Johansen and P. A. Mello, Phys. Lett. A **372**, 5760 (2008).

[15] A. Kalev and P. A. Mello, J. Phys. A: Math. Gen. **45**, 235301 (2012).

[16] J. Schwinger, Proc. Nat. Acad. Sci. USA **46**, 570 (1960).

[17] An important side question is then: what physical operations do the remaining $d^2 - 4$ generators represent? If the angular momentum is obtained adding several spin 1/2 systems, we know that these additional operations are individual spin rotations and entangling transformations. This consideration implies that all elementary particles are either spin 1/2 fermions, spin 0 bosons, or spin 1 massless bosons.

[18] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Phys. Rev. Lett. **73**, 58 (1994).

[19] Strictly speaking, it is a complex euclidean space, but we shall follow the physicist's usage that, if it has to do with quantum mechanics, then we call it a Hilbert space.